

DOUBLES FOR MONOIDAL CATEGORIES

CRAIG PASTRO AND ROSS STREET

Dedicated to Walter Tholen on his 60th birthday

ABSTRACT. In a recent paper, Daisuke Tambara defined two-sided actions on an endomodule (= endodistributor) of a monoidal \mathcal{V} -category \mathcal{A} . When \mathcal{A} is autonomous (= rigid = compact), he showed that the \mathcal{V} -category (that we call $\text{Tamb}(\mathcal{A})$) of so-equipped endomodules (that we call Tambara modules) is equivalent to the monoidal centre $\mathcal{Z}[\mathcal{A}, \mathcal{V}]$ of the convolution monoidal \mathcal{V} -category $[\mathcal{A}, \mathcal{V}]$. Our paper extends these ideas somewhat. For general \mathcal{A} , we construct a promonoidal \mathcal{V} -category $\mathcal{D}\mathcal{A}$ (which we suggest should be called the double of \mathcal{A}) with an equivalence $[\mathcal{D}\mathcal{A}, \mathcal{V}] \simeq \text{Tamb}(\mathcal{A})$. When \mathcal{A} is closed, we define strong (respectively, left strong) Tambara modules and show that these constitute a \mathcal{V} -category $\text{Tamb}_s(\mathcal{A})$ (respectively, $\text{Tamb}_{ls}(\mathcal{A})$) which is equivalent to the centre (respectively, lax centre) of $[\mathcal{A}, \mathcal{V}]$. We construct localizations $\mathcal{D}_s\mathcal{A}$ and $\mathcal{D}_{ls}\mathcal{A}$ of $\mathcal{D}\mathcal{A}$ such that there are equivalences $\text{Tamb}_s(\mathcal{A}) \simeq [\mathcal{D}_s\mathcal{A}, \mathcal{V}]$ and $\text{Tamb}_{ls}(\mathcal{A}) \simeq [\mathcal{D}_{ls}\mathcal{A}, \mathcal{V}]$. When \mathcal{A} is autonomous, every Tambara module is strong; this implies an equivalence $\mathcal{Z}[\mathcal{A}, \mathcal{V}] \simeq [\mathcal{D}\mathcal{A}, \mathcal{V}]$.

1. INTRODUCTION

For \mathcal{V} -categories \mathcal{A} and \mathcal{B} , a *module* $T : \mathcal{A} \dashrightarrow \mathcal{B}$ (also called “bimodule”, “profunctor”, and “distributor”) is a \mathcal{V} -functor $T : \mathcal{B}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$. For a monoidal \mathcal{V} -category \mathcal{A} , Tambara [Tam06] defined two-sided actions α of \mathcal{A} on an endomodule $T : \mathcal{A} \dashrightarrow \mathcal{A}$. When \mathcal{A} is autonomous (also called “rigid” or “compact”) he showed that the \mathcal{V} -category $\text{Tamb}(\mathcal{A})$ of Tambara modules (T, α) is equivalent to the monoidal centre $\mathcal{Z}[\mathcal{A}, \mathcal{V}]$ of the convolution monoidal \mathcal{V} -category $[\mathcal{A}, \mathcal{V}]$.

Our paper extends these ideas in four ways:

- (1) our base monoidal category \mathcal{V} is quite general (as in [Kel82]) not just vector spaces;
- (2) our results are mainly for a closed monoidal \mathcal{V} -category \mathcal{A} , generalizing the autonomous case;
- (3) we show the connection with the lax centre as well as the centre; and,
- (4) we introduce the double $\mathcal{D}\mathcal{A}$ of a monoidal \mathcal{V} -category \mathcal{A} and some localizations of it, and relate these to Tambara modules.

Date: February 2, 2008.

Key words and phrases. monoidal centre, Drinfeld double, monoidal category, Day convolution.

The first author gratefully acknowledges support of an international Macquarie University Research Scholarship while the second gratefully acknowledges support of the Australian Research Council Discovery Grant DP0771252.

Our principal goal is to give conditions under which the centre and lax centre of a \mathcal{V} -valued \mathcal{V} -functor monoidal \mathcal{V} -category is again such. Some results in this direction can be found in [DS07].

For general monoidal \mathcal{A} , we construct a promonoidal \mathcal{V} -category $\mathcal{D}\mathcal{A}$ with an equivalence $[\mathcal{D}\mathcal{A}, \mathcal{V}] \simeq \text{Tamb}(\mathcal{A})$. When \mathcal{A} is closed, we define when a Tambara module is (left) strong and show that these constitute a \mathcal{V} -category ($\text{Tamb}_{ls}(\mathcal{A})$) $\text{Tamb}_s(\mathcal{A})$ which is equivalent to the (lax) centre of $[\mathcal{A}, \mathcal{V}]$. We construct localizations $\mathcal{D}_s\mathcal{A}$ and $\mathcal{D}_{ls}\mathcal{A}$ of $\mathcal{D}\mathcal{A}$ such that there are equivalences $\text{Tamb}_s(\mathcal{A}) \simeq [\mathcal{D}_s\mathcal{A}, \mathcal{V}]$ and $\text{Tamb}_{ls}(\mathcal{A}) \simeq [\mathcal{D}_{ls}\mathcal{A}, \mathcal{V}]$. When \mathcal{A} is autonomous, every Tambara module is strong, which implies an equivalence $\mathcal{Z}[\mathcal{A}, \mathcal{V}] \simeq [\mathcal{D}\mathcal{A}, \mathcal{V}]$. These results should be compared with those of [DS07] where the lax centre of $[\mathcal{A}, \mathcal{V}]$ is shown generally to be a full sub- \mathcal{V} -category of a functor \mathcal{V} -category $[\mathcal{A}_M, \mathcal{V}]$ which also becomes an equivalence $\mathcal{Z}[\mathcal{A}, \mathcal{V}] \simeq [\mathcal{A}_M, \mathcal{V}]$ when \mathcal{A} is autonomous.

As we were completing this paper, Ignacio Lopez Franco sent us his preprint [LF07] which has some results in common with ours. As an example for \mathcal{V} -modules of his general constructions on pseudomonoids, he is also led to what we call the double monad.

2. CENTRES AND CONVOLUTION

We work with categories enriched in a base monoidal category \mathcal{V} as used by Kelly [Kel82]. It is symmetric, closed, complete and cocomplete.

Let \mathcal{A} denote a closed monoidal \mathcal{V} -category. We denote the tensor product by $A \otimes B$ and the unit by I in the hope that this will cause no confusion with the same symbols used for the base \mathcal{V} itself. We have \mathcal{V} -natural isomorphisms

$$\mathcal{A}(A, {}^B C) \cong \mathcal{A}(A \otimes B, C) \cong \mathcal{A}(B, C^A)$$

defined by evaluation and coevaluation morphisms

$$\begin{aligned} e_l : {}^B C \otimes B &\longrightarrow C, & d_l : A &\longrightarrow {}^B(A \otimes B), \\ e_r : A \otimes C^A &\longrightarrow C, & d_r : B &\longrightarrow (A \otimes B)^A. \end{aligned}$$

Consequently, there are canonical isomorphisms

$${}^{A \otimes B} C \cong {}^A({}^B C), \quad C^{A \otimes B} \cong (C^A)^B, \quad ({}^B C)^A \cong {}^B(C^A) \quad \text{and} \quad {}^I C \cong C \cong C^I$$

which we write as if they were identifications just as we do with the associativity and unit isomorphisms. We also write ${}^B C^A$ for ${}^B(C^A)$.

The Day convolution monoidal structure [Day70] on the \mathcal{V} -category $[\mathcal{A}, \mathcal{V}]$ of \mathcal{V} -functors from \mathcal{A} to \mathcal{V} consists of the tensor product $F * G$ and unit J defined by

$$\begin{aligned} (F * G)A &= \int^{U, V} \mathcal{A}(U \otimes V, A) \otimes FU \otimes GV \\ &\cong \int^V F({}^V A) \otimes GV \\ &\cong \int^U FU \otimes G(A^U) \end{aligned}$$

and

$$JA = \mathcal{A}(I, A).$$

In particular,

$$(F * \mathcal{A}(A, -))B \cong F({}^AB) \quad \text{and} \quad (\mathcal{A}(A, -) * G)B \cong G(B^A).$$

The centre of a monoidal category was defined in [JS91] and the lax centre was defined, for example, in [DPS07]. Since the representables are dense in $[\mathcal{A}, \mathcal{V}]$, an object of the *lax centre* $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$ of $[\mathcal{A}, \mathcal{V}]$ is a pair (F, θ) consisting of $F \in [\mathcal{A}, \mathcal{V}]$ and a \mathcal{V} -natural family θ of morphisms

$$\theta_{A,B} : F({}^AB) \longrightarrow F(B^A)$$

such that the diagrams

$$\begin{array}{ccc} F({}^{A \otimes B}C) & \xrightarrow{\theta_{A \otimes B, C}} & F(C^{A \otimes B}) \\ \downarrow = & & \uparrow = \\ F({}^A({}^BC)) & & F((C^A)^B) \\ \searrow \theta_{A, BC} & & \nearrow \theta_{B, CA} \\ & F({}^{BC}A) & \end{array} \quad \text{and} \quad \begin{array}{ccc} F({}^IA) & \xrightarrow{\theta_{I, A}} & F(A^I) \\ \searrow = & & \nearrow = \\ & FA & \end{array}$$

commute. The hom object $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]((F, \theta), (G, \phi))$ is defined to be the equalizer of two obvious morphisms out of $[\mathcal{A}, \mathcal{V}](F, G)$. The *centre* $\mathcal{Z}[\mathcal{A}, \mathcal{V}]$ of $[\mathcal{A}, \mathcal{V}]$ is the full sub- \mathcal{V} -category of $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$ consisting of those objects (F, θ) with θ invertible.

3. TAMBARA MODULES

Let \mathcal{A} denote a monoidal \mathcal{V} -category. We do not need \mathcal{A} to be closed for the definition of Tambara module although we will require this restriction again later.

A *left Tambara module* on \mathcal{A} is a \mathcal{V} -functor $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ together with a family of morphisms

$$\alpha_l(A, X, Y) : T(X, Y) \longrightarrow T(A \otimes X, A \otimes Y)$$

which are \mathcal{V} -natural in each of the objects A, X and Y , satisfying the two equations $\alpha_l(I, X, Y) = 1_{T(X, Y)}$ and

$$\begin{array}{ccc} T(X, Y) & \xrightarrow{\alpha_l(A', X, Y)} & T(A' \otimes X, A' \otimes Y) \\ \searrow \alpha_l(A \otimes A', X, Y) & & \swarrow \alpha_l(A, A' \otimes X, A' \otimes Y) \\ & T(A \otimes A' \otimes X, A \otimes A' \otimes Y). & \end{array}$$

Similarly, a *right Tambara module* on \mathcal{A} is a \mathcal{V} -functor $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ together with a family of morphisms

$$\alpha_r(B, X, Y) : T(X, Y) \longrightarrow T(X \otimes B, Y \otimes B)$$

which are \mathcal{V} -natural in each of the objects B , X and Y , satisfying the two equations $\alpha_r(I, X, Y) = 1_{T(X, Y)}$ and

$$\begin{array}{ccc} T(X, Y) & \xrightarrow{\alpha_r(B, X, Y)} & T(X \otimes B, Y \otimes B) \\ & \searrow \alpha_r(B \otimes B', X, Y) & \swarrow \alpha_r(B', B \otimes X, B \otimes Y) \\ & T(X \otimes B \otimes B', Y \otimes B \otimes B'). \end{array}$$

A *Tambara module* (T, α) on \mathcal{A} is a \mathcal{V} -functor $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ together with both left and right Tambara module structures satisfying the “bimodule” compatibility condition

$$\begin{array}{ccc} T(X, Y) & \xrightarrow{\alpha_l(A, X, Y)} & T(A \otimes X, A \otimes Y) \\ \alpha_r(B, X, Y) \downarrow & & \downarrow \alpha_r(B, A \otimes X, A \otimes Y) \\ T(X \otimes B, Y \otimes B) & \xrightarrow{\alpha_l(A, X \otimes B, Y \otimes B)} & T(A \otimes X \otimes B, A \otimes Y \otimes B). \end{array}$$

The morphism defined to be the diagonal of the last square is denoted by

$$\alpha(A, B, X, Y) : T(X, Y) \longrightarrow T(A \otimes X \otimes B, A \otimes Y \otimes B)$$

and we can express a Tambara module structure purely in terms of this, however, we need to refer to the left and right structures below.

Proposition 3.1. *Suppose \mathcal{A} is a monoidal \mathcal{V} -category and $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ is a \mathcal{V} -functor.*

- (a) *If \mathcal{A} is right closed, there is a bijection between \mathcal{V} -natural families of morphisms*

$$\alpha_l(A, X, Y) : T(X, Y) \longrightarrow T(A \otimes X, A \otimes Y)$$

and \mathcal{V} -natural families of morphisms

$$\beta_l(A, X, Y) : T(X, Y^A) \longrightarrow T(A \otimes X, Y).$$

- (b) *Under the bijection of (a), the family α_l is a left Tambara structure if and only if the family β_l satisfies the two equations $\beta_l(I, X, Y) = 1_{T(X, Y)}$ and*

$$\begin{array}{ccc} T(X, Y^{A \otimes A'}) & \xrightarrow{\beta_l(A \otimes A', X, Y)} & T(A \otimes A' \otimes X, Y) \\ \downarrow = & & \uparrow \beta_l(A, A' \otimes X, Y) \\ T(X, (Y^A)^{A'}) & \xrightarrow{\beta_l(A', X, Y^A)} & T(A' \otimes X, Y^A). \end{array}$$

- (c) *If \mathcal{A} is left closed, there is a bijection between \mathcal{V} -natural families of morphisms*

$$\alpha_r(B, X, Y) : T(X, Y) \longrightarrow T(X \otimes B, Y \otimes B)$$

and \mathcal{V} -natural families of morphisms

$$\beta_r(B, X, Y) : T(X, {}^B Y) \longrightarrow T(X \otimes B, Y).$$

- (d) Under the bijection of (c), the family α_r is a right Tambara structure if and only if the family β_r satisfies the two equations $\beta_r(I, X, Y) = 1_T(X, Y)$ and

$$\begin{array}{ccc} T(X, {}^B B' Y) & \xrightarrow{\beta_r(B \otimes B', X, Y)} & T(X \otimes B \otimes B', Y) \\ \downarrow = & & \uparrow \beta_r(B', X \otimes B, Y) \\ T(X, {}^B (B' Y)) & \xrightarrow{\beta_r(B, X, {}^B Y)} & T(X \otimes B, {}^B Y). \end{array}$$

- (e) If \mathcal{A} is closed, the families α_l and α_r form a Tambara module structure if and only if the families β_l and β_r , corresponding under (a) and (c), satisfy the condition

$$\begin{array}{ccc} T(X, {}^B Y^A) & \xrightarrow{\beta_l(A, X, {}^B Y)} & T(A \otimes X, {}^B Y) \\ \downarrow \beta_r(B, X, Y^A) & & \downarrow \beta_r(B, A \otimes X, Y) \\ T(X \otimes B, Y^A) & \xrightarrow{\beta_l(A, X \otimes B, Y)} & T(A \otimes X \otimes B, Y). \end{array}$$

Proof. The bijection of (a) is defined by the formulas

$$\beta_l(A, X, Y) = \left(T(X, Y^A) \xrightarrow{\alpha_l(A, X, Y^A)} T(A \otimes X, A \otimes Y^A) \xrightarrow{T(A \otimes X, e_r)} T(A \otimes X, Y) \right)$$

and

$$\alpha_l(A, X, Y) = \left(T(X, Y) \xrightarrow{T(X, d_r)} T(X, (A \otimes Y)^A) \xrightarrow{\beta_l(A, X, A \otimes Y)} T(A \otimes X, A \otimes Y) \right).$$

That the processes are mutually inverse uses the adjunction identities on the morphisms e and d . The bijection of (c) is obtained dually by reversing the tensor product. Translation of the conditions from the α to the β as required for (b), (d) and (e) is straightforward. \square

A left (respectively, right) Tambara module T on \mathcal{A} will be called *strong* when the morphisms

$$\begin{aligned} \beta_l(A, X, Y) : T(X, Y^A) &\longrightarrow T(A \otimes X, Y) \\ \text{(respectively, } \beta_r(B, X, Y) : T(X, {}^B Y) &\longrightarrow T(X \otimes B, Y)) \end{aligned}$$

corresponding via Proposition 3.1 to the left (respectively, right) Tambara structure, are invertible. A Tambara module is called *left* (respectively, *right*) *strong* when it is strong as a left (respectively, right) module and *strong* when it is both left and right strong. In particular, notice that the hom \mathcal{V} -functor (= identity module) of \mathcal{A} is a strong Tambara module.

Proposition 3.2. *Suppose \mathcal{A} is a monoidal \mathcal{V} -category and $T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ is a \mathcal{V} -functor. If \mathcal{A} is right (left) autonomous then every left (right) Tambara module is strong.*

Proof. If A^* denotes a right dual for A with unit $\eta : I \longrightarrow A^* \otimes A$ then an inverse for β_l is defined by the composite

$$T(A \otimes X, Y) \xrightarrow{\alpha_l(A^*, A \otimes X, Y)} T(A^* \otimes A \otimes X, A^* \otimes Y) \xrightarrow{T(\eta, 1)} T(X, A^* \otimes Y) .$$

□

Write $\text{LTamb}(\mathcal{A})$ for the \mathcal{V} -category whose objects are left Tambara modules $T = (T, \alpha_l)$ and whose hom $\text{LTamb}(\mathcal{A})(T, T')$ in \mathcal{V} is defined to be the intersection over all A, X and Y of the equalizers of the pairs of morphisms:

$$[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}](T, T') \xrightleftharpoons[\mathcal{V}(1, \alpha_l) \circ \text{pr}_{X, Y}]{\mathcal{V}(\alpha_l, 1) \circ \text{pr}_{A \otimes X, A \otimes Y}} \mathcal{V}(T(X, Y), T'(A \otimes X, A \otimes Y)) .$$

Equivalently, we can define the hom as an intersection of equalizers of pairs of morphisms:

$$[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}](T, T') \xrightleftharpoons[\mathcal{V}(1, \beta_l) \circ \text{pr}_{X, Y^A}]{\mathcal{V}(\beta_l, 1) \circ \text{pr}_{A \otimes X, Y}} \mathcal{V}(T(X, Y^A), T'(A \otimes X, Y)) .$$

Composition is defined so that we have a \mathcal{V} -functor $\iota : \text{LTamb}(\mathcal{A}) \longrightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$ which forgets the left module structure on T . In fact, $\text{LTamb}(\mathcal{A})$ becomes a monoidal \mathcal{V} -category in such a way that the forgetful \mathcal{V} -functor ι becomes strong monoidal. For this, the monoidal structure on $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$ is the usual tensor product (= composition) of endomorphisms:

$$(T \otimes_{\mathcal{A}} T')(X, Y) = \int^Z T(X, Z) \otimes T'(Z, Y) .$$

When T and T' are left Tambara modules, the left Tambara structure

$$(T \otimes_{\mathcal{A}} T')(X, Y) \longrightarrow (T \otimes_{\mathcal{A}} T')(A \otimes X, A \otimes Y)$$

on $T \otimes_{\mathcal{A}} T'$ is defined by taking its composite with the coprojection copr_Z into the above coend to be the composite

$$\begin{aligned} T(X, Z) \otimes T'(Z, Y) &\xrightarrow{\alpha_l \otimes \alpha_l} T(A \otimes X, A \otimes Z) \otimes T'(A \otimes Z, A \otimes Y) \\ &\xrightarrow{\text{copr}_{A \otimes Z}} (T \otimes_{\mathcal{A}} T')(A \otimes X, A \otimes Y) . \end{aligned}$$

Similarly we obtain monoidal \mathcal{V} -categories $\text{RTamb}(\mathcal{A})$ and $\text{Tamb}(\mathcal{A})$ of right Tambara and all Tambara modules on \mathcal{A} .

We write $\text{LTamb}_s(\mathcal{A})$ for the full sub- \mathcal{V} -category of $\text{LTamb}(\mathcal{A})$ consisting of the strong left Tambara modules. We write $\text{Tamb}_{ls}(\mathcal{A})$, $\text{Tamb}_{rs}(\mathcal{A})$ and $\text{Tamb}_s(\mathcal{A})$ for the full sub- \mathcal{V} -categories of $\text{Tamb}(\mathcal{A})$ consisting of the left strong, right strong and strong Tambara modules respectively.

If \mathcal{A} is autonomous then $\text{Tamb}(\mathcal{A}) = \text{Tamb}_{ls}(\mathcal{A}) = \text{Tamb}_{rs}(\mathcal{A}) = \text{Tamb}_s(\mathcal{A})$ by Proposition 3.2.

4. THE CAYLEY FUNCTOR

Consider a right closed monoidal \mathcal{V} -category \mathcal{A} . There is a *Cayley \mathcal{V} -functor*

$$\Upsilon : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$$

defined as follows. To each object $F \in [\mathcal{A}, \mathcal{V}]$, define $\Upsilon(F) = T_F$ by

$$T_F(X, Y) = F(Y^X).$$

The effect $\Upsilon_{F,G} : [\mathcal{A}, \mathcal{V}](F, G) \longrightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}](T_F, T_G)$ of Υ on homs is defined by taking its composite with the projection

$$\text{pr}_{X,Y} : [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}](T_F, T_G) \longrightarrow \mathcal{V}(F(Y^X), G(Y^X))$$

to be the projection

$$\text{pr}_{Y^X} : [\mathcal{A}, \mathcal{V}](F, G) \longrightarrow \mathcal{V}(F(Y^X), G(Y^X)).$$

Proposition 4.1. *The Cayley \mathcal{V} -functor Υ is strong monoidal; it takes Day convolution to composition of endomodules.*

Proof. We have the calculation:

$$\begin{aligned} (\Upsilon(F) \otimes_{\mathcal{A}} \Upsilon(G))(X, Y) &= \int^Z \Upsilon(F)(X, Z) \otimes \Upsilon(G)(Z, Y) \\ &= \int^Z F(Z^X) \otimes G(Y^Z) \\ &\cong \int^{Z,U,V} \mathcal{A}(U, Z^X) \otimes FU \otimes \mathcal{A}(V, Y^Z) \otimes GV \\ &\cong \int^{Z,U,V} \mathcal{A}(X \otimes U, Z) \otimes FU \otimes \mathcal{A}(Z \otimes V, Y) \otimes GV \\ &\cong \int^{U,V} \mathcal{A}(X \otimes U \otimes V, Y) \otimes FU \otimes GV \\ &\cong \int^{U,V} \mathcal{A}(U \otimes V, Y^X) \otimes FU \otimes GV \\ &\cong \Upsilon(F * G)(X, Y), \end{aligned}$$

and of course $\Upsilon(\mathcal{A}(I, -))(X, Y) = \mathcal{A}(I, Y^X) \cong \mathcal{A}(X, Y)$. \square

In fact, Υ lands in the left Tambara modules by defining, for each $F \in [\mathcal{A}, \mathcal{V}]$, the structure

$$\alpha_l(A, X, Y) = \left(F(Y^X) \xrightarrow{F((d_r)^X)} F((A \otimes Y)^{A \otimes X}) \right)$$

on T_F . It is helpful to observe that the β_l corresponding to this α_l (via Proposition 3.1) is given by the identity

$$\beta_l(A, X, Y) = \left(F(Y^{A \otimes X}) \xrightarrow{1} F(Y^{A \otimes X}) \right),$$

showing that T_F becomes a strong left module. To see that there is a \mathcal{V} -functor $\hat{\Upsilon} : [\mathcal{A}, \mathcal{V}] \longrightarrow \text{LTamb}_s(\mathcal{A})$ satisfying $\iota \circ \hat{\Upsilon} = \Upsilon$, we merely observe that

$$\text{pr}_{A \otimes X, Y} \circ \Upsilon_{F,G} = \text{pr}_{Y^{A \otimes X}} = \text{pr}_{(Y^A)^X} = \text{pr}_{X, Y^A} \circ \Upsilon_{F,G}.$$

Proposition 4.2. *If \mathcal{A} is a right closed monoidal \mathcal{V} -category then the \mathcal{V} -functor $\hat{\Upsilon} : [\mathcal{A}, \mathcal{V}] \longrightarrow \text{LTamb}_s(\mathcal{A})$ is an equivalence.*

Proof. Define $\zeta : \text{LTamb}(\mathcal{A})(T_F, T_G) \longrightarrow [\mathcal{A}, \mathcal{V}](F, G)$ by $\text{pr}_Y \circ \zeta = \text{pr}_{I,Y} \circ \iota_{T_F, T_G}$. Then

$$\text{pr}_Y \circ \zeta \circ \hat{\Upsilon}_{F,G} = \text{pr}_{I,Y} \circ \iota_{T_F, T_G} \circ \hat{\Upsilon}_{F,G} = \text{pr}_{I,Y} \circ \Upsilon_{F,G} = \text{pr}_Y$$

and

$$\begin{aligned} \text{pr}_{X,Y} \circ \iota_{T_F, T_G} \circ \hat{\Upsilon}_{F,G} \circ \zeta &= \text{pr}_{X,Y} \circ \Upsilon_{F,G} \circ \zeta \\ &= \text{pr}_{Y^X} \circ \zeta \\ &= \text{pr}_{I,Y^X} \circ \iota_{T_F, T_G} \\ &= \text{pr}_{X,Y} \circ \iota_{T_F, T_G}. \end{aligned}$$

It follows that ζ is the inverse of $\hat{\Upsilon}_{F,G}$, so that $\hat{\Upsilon}$ is fully faithful. To see that $\hat{\Upsilon}$ is essentially surjective on objects, take a strong left module T . Put $FY = T(I, Y)$ as a \mathcal{V} -functor in Y . Then the isomorphism $\beta_l(X, I, Y)$ yields

$$T_F(X, Y) = F(Y^X) = T(I, Y^X) \cong T(X, Y);$$

so $\hat{\Upsilon}(F) \cong T$. \square

Now suppose we have an object (F, θ) of the lax centre $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$ of $[\mathcal{A}, \mathcal{V}]$. Then T_F becomes a right Tambara module by defining

$$\alpha_r(B, X, Y) = \left(F(Y^X) \xrightarrow{F((d_l)^X)} F(B(Y \otimes B)^X) \xrightarrow{\theta_{B, (Y \otimes B)^X}} F(Y \otimes B)^{X \otimes B} \right).$$

If \mathcal{A} is left closed, the β_r corresponding to this α_r (via Proposition 3.1) is defined by

$$\beta_r(B, X, Y) = \left(F(BY^X) \xrightarrow{\theta_{B, Y^X}} F(Y^{X \otimes B}) \right).$$

It is easy to see that, in this way, $T_F = \hat{\Upsilon}(F)$ actually becomes a (two-sided) Tambara module which we write as $\hat{\Upsilon}(F, \theta)$, and we have a \mathcal{V} -functor

$$\hat{\Upsilon} : \mathcal{Z}_l[\mathcal{A}, \mathcal{V}] \longrightarrow \text{Tamb}_{ls}(\mathcal{A}).$$

Proposition 4.3. *If \mathcal{A} is a closed monoidal \mathcal{V} -category then the \mathcal{V} -functor*

$$\hat{\Upsilon} : \mathcal{Z}_l[\mathcal{A}, \mathcal{V}] \longrightarrow \text{Tamb}_{ls}(\mathcal{A})$$

is an equivalence which restricts to an equivalence

$$\hat{\Upsilon} : \mathcal{Z}[\mathcal{A}, \mathcal{V}] \longrightarrow \text{Tamb}_s(\mathcal{A}).$$

Proof. The proof of full faithfulness proceeds along the lines of the beginning of the proof of Proposition 4.2. For essential surjectivity on objects, take a left strong Tambara module (T, α) . Then $\beta_l(A, X, Y) : T(X, Y^A) \longrightarrow T(A \otimes X, Y)$ is invertible. Define the \mathcal{V} -functor $F : \mathcal{A} \longrightarrow \mathcal{V}$ by $FX = T(I, X)$ as in the proof of Proposition 4.2, and define $\theta_{A,Y} : F(A^Y) \longrightarrow F(Y^A)$ to be the composite

$$T(I, A^Y) \xrightarrow{\beta_r(A, I, Y)} T(A, Y) \xrightarrow{\beta_l(A, I, Y)^{-1}} T(I, Y^A).$$

This is easily seen to yield an object (F, θ) of the lax centre $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$ with $\hat{\Upsilon}(F, \theta) \cong T_F$. Thus we have the first equivalence. Clearly θ is invertible if and only if β_r is; the second equivalence follows. \square

5. THE DOUBLE MONAD

Tambara modules are actually Eilenberg-Moore coalgebras for a fairly obvious comonad on $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$. We begin by looking at the case of left modules.

Let $\Theta_l : [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$ be the \mathcal{V} -functor defined by the end

$$\Theta_l(T)(X, Y) = \int_A T(A \otimes X, A \otimes Y).$$

There is a \mathcal{V} -natural family $\epsilon_T : \Theta_l(T) \longrightarrow T$ defined by the projections

$$\text{pr}_I : \int_A T(A \otimes X, A \otimes Y) \longrightarrow T(X, Y).$$

There is a \mathcal{V} -natural family $\delta_T : \Theta_l(T) \longrightarrow \Theta_l(\Theta_l(T))$ defined by taking its composite with the projection

$$\text{pr}_{B,C} : \int_{B,C} T(B \otimes C \otimes X, B \otimes C \otimes Y) \longrightarrow T(B \otimes C \otimes X, B \otimes C \otimes Y)$$

to be the projection

$$\text{pr}_{B \otimes C} : \int_A T(A \otimes X, A \otimes Y) \longrightarrow T(B \otimes C \otimes X, B \otimes C \otimes Y).$$

It is now easily checked that $\Theta_l = (\Theta_l, \delta, \epsilon)$ is a comonad on $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$.

There are also a comonad Θ_r on $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$, a distributive law $\Theta_r \Theta_l \cong \Theta_l \Theta_r$, and a comonad $\Theta = \Theta_r \Theta_l$:

$$\Theta_r(T)(X, Y) = \int_B T(X \otimes B, Y \otimes B)$$

and

$$\Theta(T)(X, Y) = \int_{A,B} T(A \otimes X \otimes B, A \otimes Y \otimes B).$$

We can easily identify the \mathcal{V} -categories of Eilenberg-Moore coalgebras for these three comonads.

Proposition 5.1. *There are isomorphisms of \mathcal{V} -categories*

- $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]^{\Theta_l} \cong \text{LTamb}(\mathcal{A})$,
- $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]^{\Theta_r} \cong \text{RTamb}(\mathcal{A})$, and
- $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]^{\Theta} \cong \text{Tamb}(\mathcal{A})$.

In fact, Θ_l , Θ_r and Θ are all monoidal comonads on $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$. For example, the structure on Θ_l is provided by the \mathcal{V} -natural transformations $\Theta_l(T) \otimes_{\mathcal{A}} \Theta_l(T') \longrightarrow \Theta_l(T \otimes_{\mathcal{A}} T')$ and $\mathcal{A}(-, -) \longrightarrow \Theta_l(\mathcal{A}(-, -))$ with components

$$(1) \quad \int_A^Z \int_A T(A \otimes X, A \otimes Z) \otimes \int_B T'(B \otimes X, B \otimes Z) \longrightarrow \int_C \int^U T(C \otimes X, U) \otimes T'(U, C \otimes Y)$$

and

$$(2) \quad \mathcal{A}(X, Y) \longrightarrow \int_A \mathcal{A}(A \otimes X, A \otimes Y)$$

defined as follows. The morphism (1) is determined by its precomposite with the coprojection copr_Z and postcomposite with the projection pr_C ; the result is defined

to be the composite

$$\begin{aligned} \int_A T(A \otimes X, A \otimes Z) \otimes \int_B T'(B \otimes X, B \otimes Z) \\ \xrightarrow{\text{pr}_C \otimes \text{pr}_C} T(C \otimes X, C \otimes Z) \otimes T'(C \otimes Z, C \otimes Y) \\ \xrightarrow{\text{copr}_{C \otimes Z}} \int^U T(C \otimes X, U) \otimes T'(U, C \otimes Y). \end{aligned}$$

The morphism (2) is simply the coprojection copr_I . It follows that $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]^{\Theta_l}$ becomes monoidal with the underlying functor becoming strong monoidal; see [Moe02] and [McC02]. Clearly we have:

Proposition 5.2. *The isomorphisms of Proposition 5.1 are monoidal.*

The next thing to observe is that Θ_l , Θ_r and Θ all have left adjoints Φ_l , Φ_r and Φ which therefore become opmonoidal monads whose \mathcal{V} -categories of Eilenberg-Moore algebras are monoidally isomorphic to $\text{LTamb}(\mathcal{A})$, $\text{RTamb}(\mathcal{A})$ and $\text{Tamb}(\mathcal{A})$, respectively. Straightforward applications of the Yoneda Lemma, show that the formulas for these adjoints are

$$\begin{aligned} \Phi_l(S)(U, V) &= \int^{A, X, Y} \mathcal{A}(U, A \otimes X) \otimes \mathcal{A}(A \otimes Y, V) \otimes S(X, Y), \\ \Phi_r(S)(U, V) &= \int^{B, X, Y} \mathcal{A}(U, X \otimes B) \otimes \mathcal{A}(Y \otimes B, V) \otimes S(X, Y), \quad \text{and} \\ \Phi(S)(U, V) &= \int^{A, B, X, Y} \mathcal{A}(U, A \otimes X \otimes B) \otimes \mathcal{A}(A \otimes Y \otimes B, V) \otimes S(X, Y). \end{aligned}$$

Recall that left adjoint \mathcal{V} -functors $\Psi : [\mathcal{X}^{\text{op}}, \mathcal{V}] \longrightarrow [\mathcal{Y}^{\text{op}}, \mathcal{V}]$ are equivalent to \mathcal{V} -functors $\check{\Psi} : \mathcal{Y}^{\text{op}} \otimes \mathcal{X} \longrightarrow \mathcal{V}$, which are also called modules $\check{\Psi} : \mathcal{X} \dashrightarrow \mathcal{Y}$ from \mathcal{X} to \mathcal{Y} . The equivalence is defined by:

$$\check{\Psi}(Y, X) = \Psi(\mathcal{X}(-, X))(Y)$$

and

$$\Psi(M)(Y) = \int^X \check{\Psi}(Y, X) \otimes M(X).$$

It follows that Φ_l , Φ_r and Φ determine monads $\check{\Phi}_l$, $\check{\Phi}_r$ and $\check{\Phi}$ on $\mathcal{A}^{\text{op}} \otimes \mathcal{A}$ in the bicategory $\mathcal{V}\text{-}\mathbf{Mod}$. The formulas are:

$$\begin{aligned} \check{\Phi}_l(X, Y, U, V) &= \int^A \mathcal{A}(U, A \otimes X) \otimes \mathcal{A}(A \otimes Y, V), \\ \check{\Phi}_r(X, Y, U, V) &= \int^B \mathcal{A}(U, X \otimes B) \otimes \mathcal{A}(Y \otimes B, V), \quad \text{and} \\ \check{\Phi}(X, Y, U, V) &= \int^{A, B} \mathcal{A}(U, A \otimes X \otimes B) \otimes \mathcal{A}(A \otimes Y \otimes B, V). \end{aligned}$$

6. DOUBLES

The bicategory $\mathcal{V}\text{-}\mathbf{Mod}$ admits the Kleisli construction for monads. Write $\mathcal{D}_l\mathcal{A}$, $\mathcal{D}_r\mathcal{A}$ and $\mathcal{D}\mathcal{A}$ for the Kleisli \mathcal{V} -categories for the monads $\check{\Phi}_l$, $\check{\Phi}_r$ and $\check{\Phi}$ on $\mathcal{A}^{\text{op}} \otimes \mathcal{A}$ in the bicategory $\mathcal{V}\text{-}\mathbf{Mod}$. We call them the *left double*, *right double* and *double*

of the monoidal \mathcal{V} -category \mathcal{A} . They all have the same objects as $\mathcal{A}^{\text{op}} \otimes \mathcal{A}$. The homs are defined by

$$\begin{aligned}\mathcal{D}_l\mathcal{A}((X, Y), (U, V)) &= \int^A \mathcal{A}(U, A \otimes X) \otimes \mathcal{A}(A \otimes Y, V), \\ \mathcal{D}_r\mathcal{A}((X, Y), (U, V)) &= \int^B \mathcal{A}(U, X \otimes B) \otimes \mathcal{A}(Y \otimes B, V), \quad \text{and} \\ \mathcal{D}\mathcal{A}((X, Y), (U, V)) &= \int^{A, B} \mathcal{A}(U, A \otimes X \otimes B) \otimes \mathcal{A}(A \otimes Y \otimes B, V).\end{aligned}$$

Proposition 6.1. *There are canonical equivalences of \mathcal{V} -categories:*

- $\Xi_l : \text{LTamb}(\mathcal{A}) \simeq [\mathcal{D}_l\mathcal{A}, \mathcal{V}]$,
- $\Xi_r : \text{RTamb}(\mathcal{A}) \simeq [\mathcal{D}_r\mathcal{A}, \mathcal{V}]$, and
- $\Xi : \text{Tamb}(\mathcal{A}) \simeq [\mathcal{D}\mathcal{A}, \mathcal{V}]$.

It follows from the main result of Day [Day70] that these doubles $\mathcal{D}_l\mathcal{A}$, $\mathcal{D}_r\mathcal{A}$ and $\mathcal{D}\mathcal{A}$ all admit promonoidal structures (P_l, J_l) , (P_r, J_r) and (P, J) for which the equivalences in Proposition 6.1 become monoidal when the right-hand sides are given the corresponding convolution structures. For example, we calculate that P_l and J_l are as follows:

$$\begin{aligned}P_l((X, Y), (U, V); (H, K)) &= (\mathcal{D}_l\mathcal{A}((X, Y), -) \otimes_{\mathcal{A}} \mathcal{D}_l\mathcal{A}((U, V), -))(H, K) \\ &= \int^{Z, A, B} \mathcal{A}(H, A \otimes X) \otimes \mathcal{A}(A \otimes Y, Z) \otimes \mathcal{A}(Z, B \otimes U) \otimes \mathcal{A}(B \otimes V, K) \\ &= \int^{A, B} \mathcal{A}(H, A \otimes X) \otimes \mathcal{A}(A \otimes Y, B \otimes U) \otimes \mathcal{A}(B \otimes V, K)\end{aligned}$$

and

$$J_l(H, K) = \mathcal{A}(H, K).$$

Furthermore, there are some special morphisms that exist in these doubles $\mathcal{D}_l\mathcal{A}$, $\mathcal{D}_r\mathcal{A}$ and $\mathcal{D}\mathcal{A}$. Let $\tilde{\alpha}_l : (X, Y) \longrightarrow (A \otimes X, A \otimes Y)$ denote the morphism in $\mathcal{D}_l\mathcal{A}$ defined by the composite

$$\begin{aligned}I &\xrightarrow{j_{A \otimes X} \otimes j_{A \otimes Y}} \mathcal{A}(A \otimes X, A \otimes X) \otimes \mathcal{A}(A \otimes Y, A \otimes Y) \\ &\xrightarrow{\text{copr}_A} \mathcal{D}_l\mathcal{A}((X, Y), (A \otimes X, A \otimes Y)).\end{aligned}$$

The \mathcal{V} -functor Ξ_l has the property that $\Xi_l(T, \alpha_l)(X, Y) = T(X, Y)$ and $\Xi_l(T, \alpha_l)(\tilde{\alpha}_l) = \alpha_l$. When \mathcal{A} is right closed, we let $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A \otimes X, Y)$ denote the morphism in $\mathcal{D}_l\mathcal{A}$ defined by the composite

$$\begin{aligned}I &\xrightarrow{j_{A \otimes X} \otimes e_r} \mathcal{A}(A \otimes X, A \otimes X) \otimes \mathcal{A}(A \otimes Y^A, Y) \\ &\xrightarrow{\text{copr}_A} \mathcal{D}_l\mathcal{A}((X, Y^A), (A \otimes X, Y)).\end{aligned}$$

Then $\Xi_l(T, \alpha_l)(\tilde{\beta}_l) = \beta_l$.

Similarly, we have the morphism $\tilde{\alpha}_r : (X, Y) \longrightarrow (X \otimes B, Y \otimes B)$ in $\mathcal{D}_r\mathcal{A}$, and also, when \mathcal{A} is left closed, the morphism $\tilde{\beta}_r : (X, {}^B Y) \longrightarrow (X \otimes B, Y)$.

There are \mathcal{V} -functors $\mathcal{D}_l\mathcal{A} \longrightarrow \mathcal{D}\mathcal{A} \longleftarrow \mathcal{D}_r\mathcal{A}$ which are the identity functions on objects and are defined on homs using projections with $B = I$ for the left leg

and the projections $A = I$ for the second leg. In this way, the morphisms $\tilde{\alpha}_l$ and $\tilde{\alpha}_r$ can be regarded also as morphisms of $\mathcal{D}\mathcal{A}$. Under closedness assumptions, the morphisms $\tilde{\beta}_l$ and $\tilde{\beta}_r$ can also be regarded as morphisms of $\mathcal{D}\mathcal{A}$.

Let Σ_l denote the set of morphisms $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A \otimes X, Y)$, let Σ_r denote the set of morphisms $\tilde{\beta}_r : (X, {}^B Y) \longrightarrow (X \otimes B, Y)$, and let Σ denote the set of morphisms $\Sigma = \Sigma_l \cup \Sigma_r$. Under appropriate closedness assumptions on \mathcal{A} , we can form various \mathcal{V} -categories of fractions such as:

- $\mathbf{L}\mathcal{D}\mathcal{A} = \mathcal{D}_l\mathcal{A}[\Sigma_l^{-1}]$ and $\mathbf{R}\mathcal{D}\mathcal{A} = \mathcal{D}_r\mathcal{A}[\Sigma_r^{-1}]$,
- $\mathcal{D}_{ls}\mathcal{A} = \mathcal{D}\mathcal{A}[\Sigma_l^{-1}]$ and $\mathcal{D}_{rs}\mathcal{A} = \mathcal{D}\mathcal{A}[\Sigma_r^{-1}]$, and
- $\mathcal{D}_s\mathcal{A} = \mathcal{D}\mathcal{A}[\Sigma^{-1}]$.

The following result is now automatic.

Theorem 6.2. *For a closed monoidal \mathcal{V} -category \mathcal{A} , there are equivalences of \mathcal{V} -categories:*

- $[\mathbf{L}\mathcal{D}\mathcal{A}, \mathcal{V}] \simeq \mathbf{LTamb}_s(\mathcal{A}) \simeq [\mathcal{A}, \mathcal{V}]$,
- $[\mathcal{D}_{ls}\mathcal{A}, \mathcal{V}] \simeq \mathbf{Tamb}_{ls}(\mathcal{A}) \simeq \mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$, and
- $[\mathcal{D}_s\mathcal{A}, \mathcal{V}] \simeq \mathbf{Tamb}_s(\mathcal{A}) \simeq \mathcal{Z}[\mathcal{A}, \mathcal{V}]$.

The first equivalence of Theorem 6.2 implies that $\mathbf{L}\mathcal{D}\mathcal{A}$ and \mathcal{A} are Morita equivalent. This begs the question of whether there is a \mathcal{V} -functor relating them more directly. Indeed there is. We have a \mathcal{V} -functor

$$\Pi : \mathcal{D}_l\mathcal{A} \longrightarrow \mathcal{A}$$

defined on objects by $\Pi(X, Y) = Y^X$ and by defining the effect

$$\Pi : \mathcal{D}_l\mathcal{A}((X, Y), (U, V)) \longrightarrow \mathcal{A}(Y^X, V^U)$$

on hom objects to have its composite with the A -coprojection equal to the composite

$$\begin{aligned} & \mathcal{A}(U, A \otimes X) \otimes \mathcal{A}(A \otimes Y, V) \\ & \xrightarrow{V^{(-)} \otimes (-)^{A \otimes X}} \mathcal{A}(V^{A \otimes X}, V^U) \otimes \mathcal{A}((A \otimes Y)^{A \otimes X}, V^{A \otimes X}) \\ & \xrightarrow{\text{composition}} \mathcal{A}((A \otimes Y)^{A \otimes X}, V^U) \\ & \xrightarrow{\mathcal{A}((d_r)^X, V^U)} \mathcal{A}(Y^X, V^U). \end{aligned}$$

It is easy to see that Π takes the morphisms $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A \otimes X, Y)$ to isomorphisms. So Π induces a \mathcal{V} -functor

$$\hat{\Pi} : \mathbf{L}\mathcal{D}_l\mathcal{A} \longrightarrow \mathcal{A};$$

this induces the first equivalence of Theorem 6.2.

For closed monoidal \mathcal{A} , the second and third equivalences of Theorem 6.2 show that both the lax centre and the centre of the convolution monoidal \mathcal{V} -category $[\mathcal{A}, \mathcal{V}]$ are again functor \mathcal{V} -categories $[\mathcal{D}_{ls}\mathcal{A}, \mathcal{V}]$ and $[\mathcal{D}_s\mathcal{A}, \mathcal{V}]$. Since $\mathcal{Z}_l[\mathcal{A}, \mathcal{V}]$ and $\mathcal{Z}[\mathcal{A}, \mathcal{V}]$ are monoidal with the tensor products colimit preserving in each variable, using the correspondence in [Day70], there are lax braided and braided promonoidal structures on $\mathcal{D}_{ls}\mathcal{A}$ and $\mathcal{D}_s\mathcal{A}$ which are such that $[\mathcal{D}_{ls}\mathcal{A}, \mathcal{V}]$ and $[\mathcal{D}_s\mathcal{A}, \mathcal{V}]$ become closed monoidal under convolution, and the equivalences of Theorem 6.2 become lax braided and braided monoidal equivalences.

Remark. We are grateful to Brian Day for pointing out that the \mathcal{V} -category \mathcal{A}_M appearing in [DS07] is equivalent to the full sub- \mathcal{V} -category of \mathcal{DA} consisting of the objects of the form (I, Y) .

He also pointed out that a consequence of Theorem 6.2 is that the centre of \mathcal{V} as a \mathcal{V} -category is equivalent to \mathcal{V} itself. This also can be seen directly by using the \mathcal{V} -naturality in X of the centre structure $u_X : A \otimes X \longrightarrow X \otimes A$ on an object A of \mathcal{V} , and the fact that $u_I = 1$, to deduce that $u_X = c_{A,X}$ (the symmetry of \mathcal{V}). Generally, the centre of \mathcal{V} as a monoidal **Set**-category is not equivalent to \mathcal{V} .

REFERENCES

- [Day70] Brian Day. On closed categories of functors, in *Reports of the Midwest Category Seminar IV*, Lecture Notes in Mathematics 137 (1970), pp. 1–38.
- [DPS07] B. Day, E. Panchadcharam, and R. Street. Lax braidings and the lax centre, *Contemporary Mathematics* 441 (2007). See <http://www.ams.org/bookstore-getitem/item=conm-441>.
- [DS07] Brian Day and Ross Street. Centres of monoidal categories of functors, *Contemporary Mathematics* 431 (2007), pp. 187–202.
- [JS91] André Joyal and Ross Street. Tortile Yang-Baxter operators in tensor categories, *J. Pure Appl. Algebra* 71 (1991), pp. 43–51.
- [Kel82] G. M. Kelly. Basic concepts of enriched category theory, London Mathematical Society Lecture Note Series 64. Cambridge University Press, Cambridge, 1982. Also at <http://www.tac.mta.ca/tac/reprints/articles/10/tr10.pdf>.
- [LF07] I. L. Lopez Franco. Hopf modules for autonomous pseudomonoids and the monoidal centre, preprint (2007).
- [McC02] Paddy McCrudden. Opmonoidal monads, *Theory Appl. Categories* 10 (2002), pp. 469–485.
- [Moe02] I. Moerdijk, Monads on tensor categories, *J. Pure Appl. Algebra* 168 no. 1-2 (2002), pp. 189–208.
- [Tam06] D. Tambara. Distributors on a tensor category, *Hokkaido Math. Journal* 35 (2006), pp. 379–425.

E-mail address: {craig, street}@maths.mq.edu.au

CENTRE OF AUSTRALIAN CATEGORY THEORY, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2109 AUSTRALIA